Exploring MATHEMATICAL PATTERNS using dynamic geometry software

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Introduction

In our experience, many students view mathematics as a collection of theorems and facts that were discovered by intelligent mathematicians and all they have left to do is to study them carefully. We believe it is important to help students change this view (Lavy & Shriki, 2003). One possible way is to enable students to modify the attributes of a given situation, and then pose problems that concern the newly generated situation (Brown & Walter, 1993). In this paper we wish to demonstrate the simplicity of such a process, and exemplify the idea that even a fundamental situation can serve as a trigger for discovering unknown patterns.

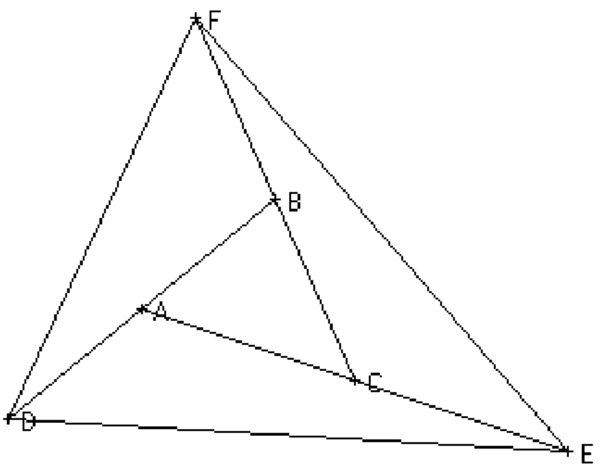

Figure 1

Revisiting triangles

Looking at a triangle, one can think of numerous questions that can be asked. We present an example of surprising and unexpected results that emerge as a consequence of one initial simple question. The answer to this question leads us to ask further related questions, and to discover patterns that we were not aware of previously.

Take a triangle ABC with an area A_0: each side of the triangle is lengthened to twice its length (BD = 2AB, AE = 2AC, CF = 2CB) and the new endpoints, D, E, F are connected to form a new triangle (Figure 1). The area of triangle DEF is designated by A_1.[1]

Question 1

Is there any connection between A_1 and A_0?

Answer 1

Using dynamic software (for example, the *Geometry Inventor* of Logal Company, Israel) it is easy to find out that $A_1/A_0 = 7$, regardless of the side lengths of triangle ABC.

1. After completing the above activity we found out that in 1997 a similar unpublished activity related only to modification 1 was developed by Ziva Levin in the framework of 'mahar 98' — workshops for mathematics teachers.

Modification 1

In this stage it seems natural to repeat the process and lengthen each side to three times its length, four times its length, and so on. Consequently we obtain triangles GHI, JKL and MNP such that

BA = AD = DG = GJ = JM;
AC = CE = EH = HK = KN and
CB = BF = FI = IL = LP (Figure 2).

The areas of triangles GHI, JKL and MNP are A_2, A_3 and A_4 respectively.

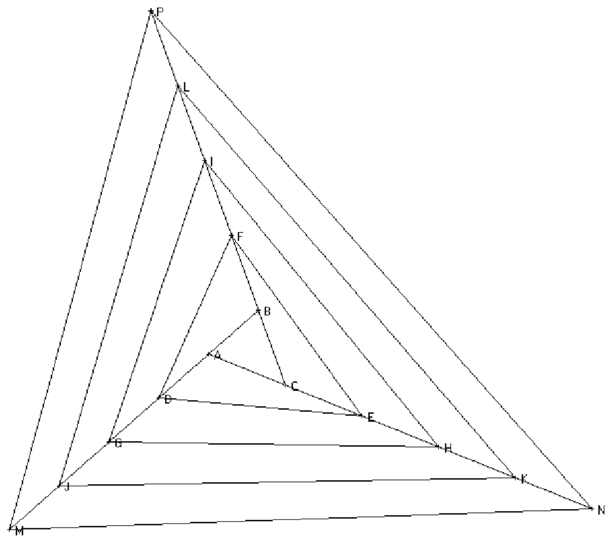

Figure 2

Question 2

Is there any connection between
$A_0, A_1, A_2, A_3, \ldots A_n$?

Answer 2

Using the software we are surprised to see that starting with any initial triangle, the ratios remain constant: $A_1/A_0 = 7$, $A_2/A_0 = 19$, $A_3/A_0 = 37$, $A_4/A_0 = 61\ldots$

Question 3

Is there any algebraic pattern that can describe the obtained ratios?

Answer 3

If the ratios are to be taken as subsequent terms of a sequence then we get:

$a_0 = 1$, $a_1 = 7$, $a_2 = 19$, $a_3 = 37$, and $a_4 = 61$.

The sequence of differences is:

$b_1 = 6$, $b_2 = 12$, $b_3 = 18$, $b_4 = 24\ldots$

which are subsequent terms of an arithmetic sequence with difference 6.
Thus, for $n = 1, 2, 3, 4\ldots$ we get:

$$\begin{aligned} a_n &= (1 + 6) + 12 + 18 + 24 + \ldots + 6n \\ &= 1 + 6 \times 1/2 \times n(n + 1) \\ &= 3n^2 + 3n + 1. \end{aligned}$$

Question 4

Can we establish this formula for a_n mathematically?

Answer 4

The answer to this question is 'Yes,' and is based on simple trigonometric considerations. Referring to Figure 3:

Let $\angle BAC = \alpha$; $\angle CBA = \beta$; $\angle ACB = \gamma$. Then:

$$\text{(1)} \quad \begin{aligned} A_0 &= \text{area } (\Delta ABC) \\ &= 1/2\ bc \sin\alpha \\ &= 1/2\ ac \sin\beta \\ &= 1/2\ ab \sin\gamma \end{aligned}$$

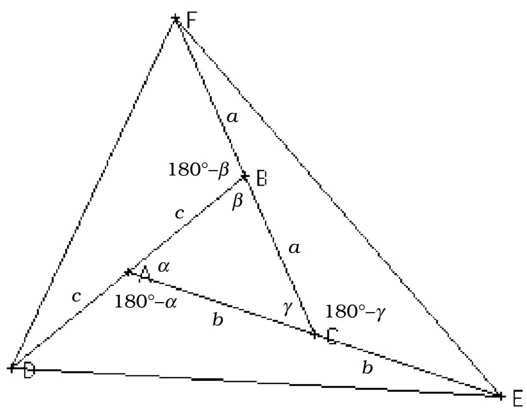

Figure 3

(2) area (ΔADE) $= 1/2\ 2bc \sin(180 - \alpha)$
$= bc \sin \alpha$
$= 2A_0$

(3) area (ΔDBF) $= 1/2\ 2ca \sin(180 - \beta)$
$= ac \sin \beta$
$= 2A_0$

(4) area (ΔEFC) $= 1/2\ 2ab \sin(180 - \gamma)$
$= ab \sin \gamma$
$= 2A_0$

From (1) – (4) we get:

A_1 = area (ΔDEF)
$= A_0 + 2A_0 + 2A_0 + 2A_0$
$= 7A_0$

so $a_1 = 7$ as required.

Similarly, for the nth triangle we get:

A_n = area (nth triangle)
$= 1/2\ nc(n + 1)b \sin \alpha$
$+ 1/2\ na(n + 1)c \sin \beta$
$+ 1/2\ nb(n + 1)a \sin \gamma + 1/2\ bc \sin \alpha$
$= 1/2\ n(n + 1)$
$[cb \sin a + ac \sin \beta + ba \sin \gamma]$
$+ 1/2\ bc \sin a$
$= 1/2\ n(n + 1)\ [2A_0 + 2A_0 + 2A_0] + A_0$
$= [3n(n + 1) + 1]\ A_0$

or $a_n = 3n^2 + 3n + 1$.

Thus we get:

$a_n = 3n^2 + 3n + 1.$

This checks for the first four cases, and we have proved the generalised case.

Note:
We have been considering terms in a sequence, so n is a natural number. Nevertheless, in the general proof we do not limit n to be a natural number. As a consequence, if we look at the result in its geometrical sense, it is clear that n can be any positive number.

We now propose a number of further extensions, giving the answers without proof. Each of these can be explored using dynamic geometry software, and then verified mathematically, with trigonometric proofs of increasing length!

Modification 2

Question 5

What connections will be discovered if we lengthen one of the sides k times its length, one of the sides l times its length and one of the sides m times its length?

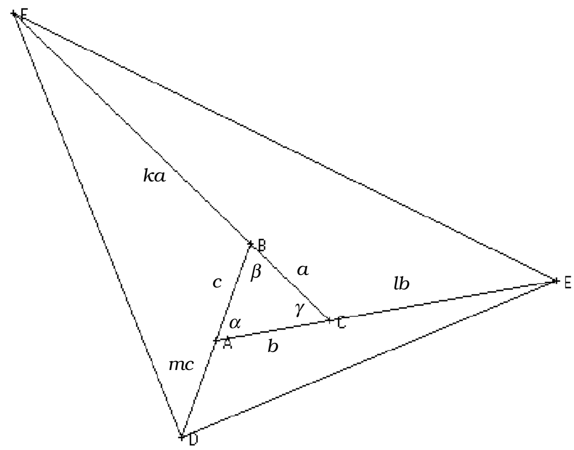

Figure 4

In Figure 4, BF = ka, CE = lb, AD = mc.

Answer 5

You can show that

$a_1 = A_1/A_0$
$= (kl + lm + mk) + (k + l + m) + 1,$

and more generally,

$a_n = A_n/A_0$
$= (kl + lm + mk)n^2 + (k + l + m)n + 1.$

Notice how nicely this extends Modification 1.

Modification 3

Question 6

The last unexpected result motivates our next question: what connections can be discovered if we replicate the process using a quadrangle in place of the initial triangle?

Answer 6

Repeating the above procedure (Figure 5), we get:

A_1 = area (EFGH) = 5 A_0,
A_2 = area (IJKL) = 13 A_0,
A_3 = area (NPQM) = 25 A_0,
A_4 = area (SVUT) = 41 A_0,

or $a_1 = 5$, $a_2 = 13$, $a_3 = 25$, $a_4 = 41$…

Arguing as in Answer 3, we now conjecture that $a_n = 2n^2 + 2n + 1$, for $n = 1, 2, 3$… and this can be established with some easy but lengthy trigonometric calculation.

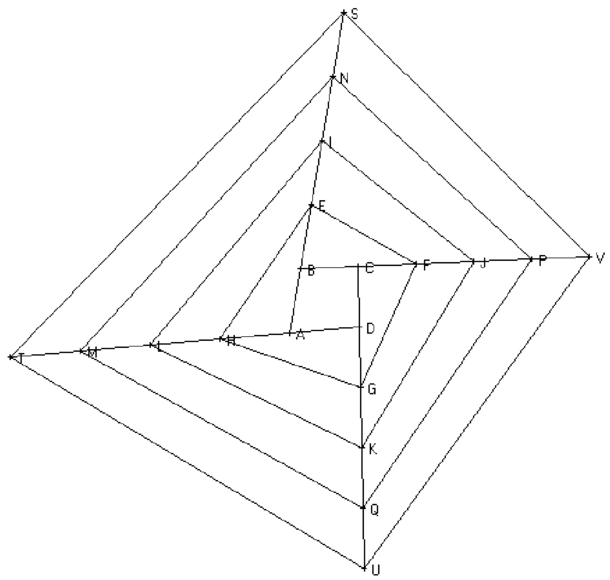

Figure 5

Modification 4

An obvious extension for the quadrangle case is the following question.

Question 7

What connections will be discovered if we lengthen one of the sides k times its length, one of the sides l times its length, one of the sides m times its length, and one of the sides p times its length?

Modification 5

In this stage we begin to develop the expectation that it would be possible to find a suitable pattern for the relations between areas for any polygon that is built in the described manner.

Question 8

What connections can be discovered if we replicate the process using a pentagon in place of the initial triangle?

Answer 8

Using the software we quickly realise that for any initial polygon with more than four sides the ratio A_1/A_0 changes as the side lengths are modified.

Question 9

We usually tend to ignore results that do not fulfil our expectations. It is no less important, however, to be able to explain why patterns can be found in certain conditions whereas they do not exist in others. The essential question hence is: why does the ratio A_n/A_0 not depend on the lengths of the sides in case of triangle and quadrangle, but does depend on their lengths in the cases of other polygons?

Answer 9

In order to answer that question, we return to the first two cases, and examine the meaning of the obtained results in a geometrical manner. Answer 4 explains why area(ΔECF) = area(ΔDAE) = area(ΔFBD) = 2area(ΔABC). As

for the quadrangle, looking at Figure 5 it is easy to verify that

area(ΔBEF) = 2area(ΔABC);
area(ΔHGD) = 2area(ΔADC);
area(ΔHAE) = 2area(ΔABD); and
area(ΔFCG) = 2area(ΔBCD).

Adding, this explains the result $A_1 = 5A_0$.

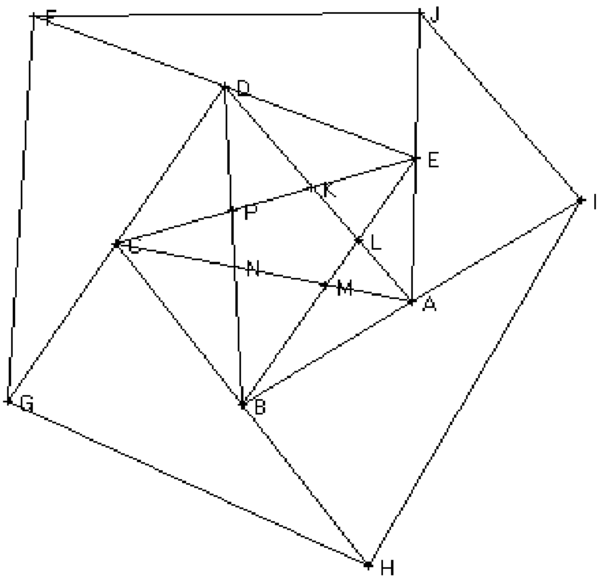

Figure 6

Relating to the pentagon in Figure 6, the difference between the cases of the triangle and quadrangle and the case of the pentagon can be seen. In this figure we can show that:

area(ΔJAI) = 2area(ΔEBA);
area(ΔBIH) = 2area(ΔABC);
area(ΔCHG) = 2area(ΔBCD);
area(ΔFDG) = 2area(ΔCDE); and
area(ΔFJE) = 2area(ΔDEA).

The area of the inner pentagon PKLMN has no connection to the triangles that are adjacent to the pentagon. Consequently the ratio A_n/A_0 depends on the pentagon's sides and angles. Note that dividing the pentagon into three triangles (using three diagonals) and using trigonometric considerations (the sine and cosine rules) it can be shown that the ratio A_n/A_0 is a function of the pentagon's sides and angles. We leave the reader to complete this proof.

Summary

In this paper we have demonstrated that posing problems can lead to some unexpected results. When students experience the joy of discovering unknown patterns, no doubt they will alter their attitude towards mathematics.

One of the National Council of Teachers of Mathematics' (2000) recommendations concerns connections:

> An emphasis on mathematical connections helps students recognise how ideas in different areas are related. Students should come both to expect and to exploit connections, using insights gained in one context to verify conjectures in another.

In our case, we began with a geometrical problem. During the process of asking questions that were based on that problem, we obtained algebraic patterns, which were proved by using trigonometric arguments. The whole process reveals the beauty that can be found in mathematics, since it exhibits connectivity among its various areas.

The whole process was enabled by the use of dynamic computer software. The possibility of easily verifying the existence of a certain phenomenon by the software motivated the search for a formal proof. Presumably without the software the whole process might never have been initiated.

References

Brown, S. I. & Walter, M. I. (1993). Problem posing in mathematics education. In S. I. Brown & M. I. Walter (Eds), *Problem Posing: Reflection and Applications* (pp. 16–27). Hillsdale, NJ: Lawrence Erlbaum.

Lavy, I. & Shriki, A. (2003). Pre-service teachers' transition from 'knowing that' to 'knowing why' via computerized project-based-learning. In N. A. Pateman, B. J. Dougherty & J. Zilliox (Eds), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education, Vol. 3* (pp. 181–188). Honolulu, HI: Center for Research and Development Group, University of Hawaii.

National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics*. Reston, VA: Author.

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